Stability and Instability of the Reissner-Nordström Cauchy Horizon and the Problem of Uniqueness in General Relativity

Mihalis Dafermos

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Abstract

This talk will describe some recent results [16] regarding the problem of uniqueness in the large (also known as *strong cosmic censorship*) for the initial value problem in general relativity. The interest in the issue of uniqueness in this context stems from its relation to the validity of the principle of determinism in classical physics. As will be clear from below, this problem does not really have an analogue in other equations of evolution typically studied. Moreover, in order to isolate the essential analytic features of the problem from the complicated setting of gravitational collapse in which it arises, some familiarity with the conformal properties of certain celebrated special solutions of the theory of relativity will have to be developed. This talk is an attempt to present precisely these features to an audience of non-specialists, in a way which hopefully will fully motivate a certain characteristic initial value problem for the spherically-symmetric Einstein-Maxwell-Scalar Field system. The considerations outlined here leading to this particular initial value problem are well known in the physics relativity community, where the problem of uniqueness has been studied heuristically [1, 22] and numerically [2, 3]. In [16], the global behavior of generic solutions to this IVP, and in particular, the issue of uniqueness, is completely understood. Only a sketch of the ideas of the proof is provided here, but the reader may refer to [16] for details.

1 General Relativity and its Initial Value Problem

The general theory of relativity is thought to provide the correct classical description for the interaction of gravity with matter. This description is embodied in a system of partial differential equations on a four dimensional manifold M, the so-called *Einstein equations*, which relate the Ricci curvature $R_{\mu\nu}$ of an

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unknown metric $g_{\mu\nu}$ to the energy-momentum tensor $T_{\mu\nu}$ of matter:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2T_{\mu\nu}. \tag{1}$$

To complete the classical picture of a physics based on a collection of fields satisfying a closed system of equations, one must also consider the laws which govern the evolution of the matter fields generating the energy-momentum tensor on the right hand side of (1). (One important special case is when there is no matter, the so-called vacuum. Then (1) with vanishing right hand side is a closed system of quasilinear hyperbolic equations.) In general, one arrives at quite complicated systems of equations. However, from the perspective of classical physics, all phenomena are in principle described by the solutions of such a system. Moreover, for these systems, the initial value problem is natural, just as in classical dynamics.

Thus, from one point of view, the general theory of relativity is a classical physical theory that can be studied mathematically in parallel with other field theories of nineteenth-century classical physics. Indeed, the equations of general relativity exhibit similar local behavior with other equations of evolution as regards, for instance, the issues of local existence and uniqueness of solutions to the initial value problem. When one turns, however, to the initial value problem in the large, the Einstein equations present features that have no analogue in other typical equations of mathematical physics. The subject of this talk will be what appears, at least at first sight, as the most pathological of these features, namely the possibility of loss of uniqueness of the solution of the initial value problem *without loss of regularity*. This possibility is at the center of what is known as the *strong cosmic censorship conjecture* formulated by Penrose [21].

The reason why the theory of the initial value problem in the large for the Einstein equations is richer than for other non-linear wave equations is that the global geometry of the characteristics is not constrained *a priori* by any other structure. This geometry, which corresponds precisely to the conformal geometry for the vacuum equations, is *a priori* unknown. It turns out that many features of the initial value problem for hyperbolic equations that one takes for granted actually depend on certain global properties of the geometry of the characteristics; the question of uniqueness indicated above is one of these.

The best way to gain some intuition for what kind of conformal geometric structure develops in the course of evolution in general relativity–and what are the implications of this structure–is to carefully examine the special solutions of the theory. In fact, almost all conjectures and intuition regarding the theory in the end derives from simple properties of such solutions. Moreover, since our focus of interest is global *geometric* structure, there is no substitute in building intuition than a good pictorial representation. This talk will rely very much on such "pictures". It should be noted, however, that in the spherically symmetric context in which we shall be working, these "pictures", besides conveying intuition, also carry complete and precise information and can be treated on the same level as symbols or formulas.

The assumption of spherical symmetry and associated pictorial representations will be carefully discussed in the next section. We will then proceed to examine a series of special solutions which will lead to a particular initial value problem. Finally, theorems describing the solutions of the initial value problem will be formulated and their proofs will be discussed.

In regard to uniqueness, it turns out that there is always a spacetime which can be uniquely associated to initial data[1]. This is the so-called *maximal domain of development* [5]. It is the "biggest" spacetime which admits the given initial hypersurface as initial data and is at the same time *globally hyperbolic*, i.e. all inextendible causal curves intersect the initial hypersurface precisely once. This latter property ensures that the domain of dependence property holds. The question of uniqueness in general relativity is thus the issue of the *extendiblity* of this maximal domain of development. If it is extendible, then the solution is not unique. Since, as noted earlier, there is really no substitute for a pictorial representation, we defer further discussion of this till later on.

2   Spherical Symmetry

The current state of affairs in the theory of quasilinear hyperbolic partial differential equations in several space variables is such that global, large data problems appear beyond reach. For there to be any hope of making headway, it seems that some sort of reduction must be made to a problem where the number of the independent variables is no more than two. For hyperbolic equations of evolution, such reductions in general are accomplished by considering symmetric solutions, or equivalently, symmetric initial data. In general relativity, symmetry assumptions are formulated in terms of a group which acts by isometry on the spacetime and preserves all matter.
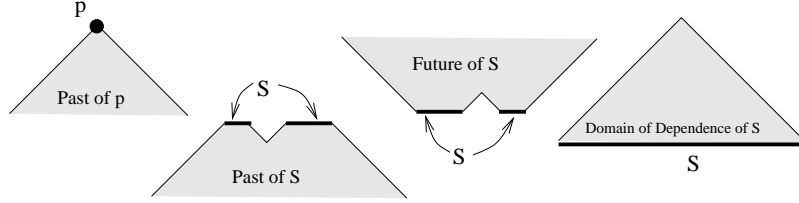
The only 2-dimensional symmetry group that is compatible with the notion of an isolated gravitating system, i.e. that can act on asymptotically flat spacetimes, is $SO(3)$. Solutions invariant under such an action are called *spherically symmetric*. As we shall see below, most of the expected phenomena of gravitational collapse of isolated gravitating systems, and the fundamental questions that these phenomena pose, can be suggested by the spherically symmetric solutions of various Einstein-matter systems. Moreover, the conformal structure of these solutions, which is the essential ingredient for the phenomena we wish to discuss, can be completely represented on the blackboard. (Or on paper!) The reason for this is simple: The space of group orbits

$$Q = M/SO(3)$$

can be given the structure of a 2-dimensional Lorentzian manifold. Restricting to Q which are *maximal domains of development* of initial data, it follows that
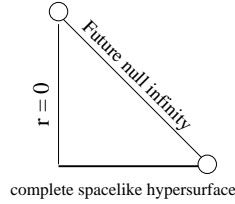
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[1]For the vacuum, initial data is a Riemannian 3-manifold $(M, \tilde{g})$, along with a symmetric 2-tensor K satisfying the constraint equations that would arise if K were to be the second fundamental form of M realized as a hypersurface in a Ricci flat 4-manifold.

these can be globally conformally represented as bounded domains in $1 + 1$-dimensional Minkowski space. The images of such representations are called *Penrose diagrams*; from these, the conformal geometry can be immediately read off as the characteristics are just the lines at $\pi/4$ or $-\pi/4$ radians from the horizontal:

3   Minkowski space

To gain some familiarity with these diagrams, it is perhaps best to begin with 4-dimensional Minkowski space from this point of view, i.e. with the Penrose diagram of the maximal domain of development of Minkowski initial data. Here the diagram is as follows:
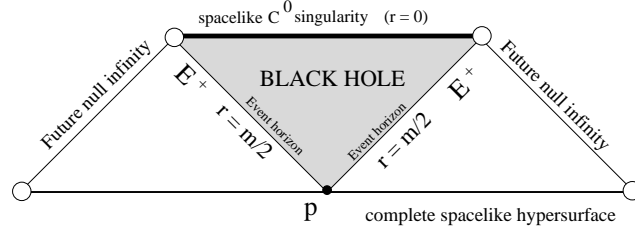

The r referred to above is a function on Q defined to be a multiple of the square root of the area of the group orbit corresponding to the points of Q. The line labelled $r = 0$ is thus the axis of symmetry. The line labelled "future null infinity" is not part of the spacetime but should be thought of as a "boundary" at infinity. The same applies to its two endpoints, "spacelike infinity" and "future timelike infinity". The latter corresponds to the "endpoint" of all inextendible future timelike geodesics.

The above Minkowski space is of course future causally geodesically complete, i.e. all causal curves can be extended to infinite affine parameter. Thus we have the analogue of global existence and uniqueness. That these properties of Minkowski space are stable to small perturbations (*without* any symmetry assumptions) is a deep theorem of Christodoulou and Klainerman [13].

4   Schwarzschild

Having understood the conformal diagram of Minkowski space, we turn to a more interesting solution: the Schwarzschild solution. This is actually a one-parameter family of solutions (the parameter is called mass and denoted by m)

which contains Minkowski space (the case where $m = 0$). As it is a spherically symmetric vacuum solution, its non-triviality in the case $m \neq 0$ must be generated by topology. Any Cauchy hypersurface has two asymptotically flat ends and topology $S^2 \times R$. "Downstairs", this corresponds to a line with two $r = \infty$ endpoints, not intersecting an axis of symmetry. For convenience, we will choose a time-symmetric initial hypersurface. The maximal development of this "Schwarzschild" initial data then looks like:

The point p depicted where $r = \frac{m}{2}$ is a minimal surface "upstairs" in the initial hypersurface. As the initial hypersurface was chosen to be time symmetric, this minimal surface is also what is called marginally trapped. The "ingoing" and "outgoing" null cones emanating from this surface, which correspond to the two null rays through p "downstairs", can thus not reach "future null infinity". (This is related to the so-called "singularity theorems" of Penrose.) Moreover, all timelike geodesics emanating from the point p reach in finite time the curve $r = 0$. This curve is not an axis of symmetry but a C^0 singularity! That is to say, there is no extension of the spacetime above through $r = 0$ with a continuous Lorentzian metric.

This singular behavior of the Schwarzschild solution may at first appear to be an undesirable feature. Indeed, historically, it was first considered exactly as such. But it turns out in fact that the behavior outlined above would provide the "ideal" scenario for the end-state of gravitational collapse. This has to do with two specific features of this solution:

1. Labelling the null rays emanating from the minimal surface as E^+, it turns out that there is a class of timelike observers, namely those who do not cross E^+, who can observe for infinite time and whose causal past is completely regular. The singularity is hidden inside a black hole, and E^+ is called the event horizon. To be more precise about the "completeness" property of the region outside the black hole, fix some outgoing null geodesic which intersects future null infinity and parallel translate a conjugate null vector (i.e. a null vector in the other direction). The affine length of the null curves joining this null geodesic and E^+, where the affine parameter is determined by the aforementioned vector, goes to ∞. In particular, future null infinity can be thought to have infinite affine length. One says that the solution pocesses a "past complete future null infinity" or a complete domain of outer communications.

2. The above spacetime is future inextendible as a C^0 metric. Thus, according to the discussion in the Introduction, this means that the Schwarzschild solution is unique even in the class of very low regularity solutions. The significance of this fact will become clear later.
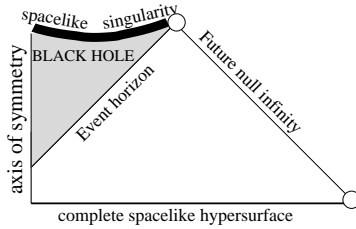
5   Christodoulou's solutions

At this point it should be noted that while the Schwarzschild solution indeed provides intuition about black holes, it cannot give insight as to whether these can occur in evolution of data where no trapped surfaces are present initially, i.e. whether the kind of behavior outlined above is related in any way to the endstate of gravitational collapse. That Properties 1 and 2 above are indeed general properties of solutions was proven by Christodoulou [6] for the spherically-symmetric Einstein scalar field equations:

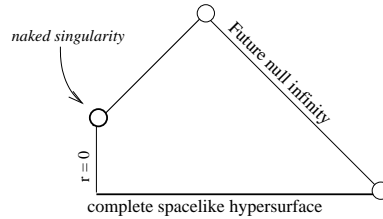$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2T_{\mu\nu},$$

$$g^{\mu\nu}(\partial_\mu \phi)_{;\nu} = 0,$$

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho \phi \partial_\sigma \phi.$$

For generic solutions of the initial value problem, the Penrose diagram obtained by Christodoulou is as follows:

In [12], however, Christodoulou explicitly constructs solutions with conformal diagram:

These are so-called "naked singularities".

One might ask why should one consider the coupling with the scalar field. The natural case to consider first, it would seem, is the vacuum. A classical

theorem of Birkhoff, however, states that the only spherically symmetric vacuum solutions are Schwarzschild. Thus, matter *must* be included to give the problem enough dynamical degrees of freedom in spherical symmetry. The scalar field is in some sense the simplest, most natural choice.[2]

As far as Property 1 is concerned, Christodoulou's results are the best evidence yet that this is indeed a property of "realistic" graviational collapse.[3] It turns out however that there is another "competing" set of evidence that indicates that the behavior of Christodoulou's solutions related to Property 2 does not represent "realistic collapse". (Remember that Property 2 is the central question for us, as this is what determines the notion of uniqueness.) This evidence is provided again by the intuition given by special solutions.

6   The Kerr and Reissner-Nordström families

One may consider the Schwarzschild family of solutions as embedded in a larger, 2-parameter family of solutions called the *Kerr solutions*. Here the parameters are called mass and angular momentum, and Schwarzschild corresponds to vanishing angular momentum. For all non-vanishing values of angular momentum, the internal structure of the black hole is completely different, and, as we shall see momentarily, much more "problematic", as Property 2 will fail. Thus the introduction of even an arbitrarily small amount of angular momentum–a phenomenon that cannot be "seen" by spherically-symmetric models–seems to change everything and cast doubt on the conclusions derived from the spherically symmetric Einstein-Scalar Field model.

To summarize our "unhappy" situation, it seems that the phenomenon which plays a fundamental role in the issue we want to study is incompatible with the assumptions we have to make in order to render it mathematically tractable. It would seem that understanding the black hole region of realistic gravitational collapse using a spherically symmetric model is a lost cause.

Fortunately, there is a "solution" to this problem! The effect of angular momentum on gravity turns out to be similar to the effect of charge. (As John Wheeler puts it, charge is a poor man's angular momentum.) Indeed, there is a very close similarity between the conformal structure of the Kerr family and a 2-parameter *spherically symmetric* family of solutions to the Einstein-Maxwell equations:
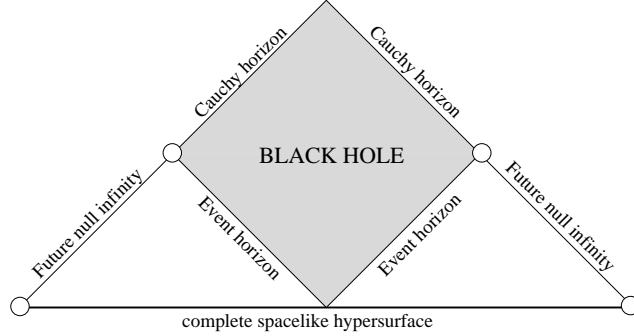
$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 2 T_{\mu\nu}$$

$$F^{\mu\nu}_{;\nu} = 0,$$

$$F_{[\mu\nu,\rho]} = 0,$$

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[2]It satisfies a linear equation and does not form singularities in the absense of coupling, it is hyperbolic so does not change the hyperbolic character of the equations, and moreover, its characteristics coincide with those of the metric g, etc. . .

[3]The statement that for generic initial data, the domain of outer communications is complete is known as "weak cosmic censorship".
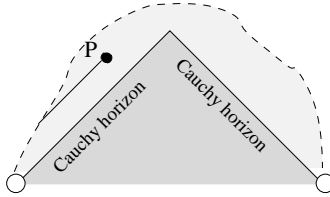
$$T\mu\nu = F_{\mu\lambda}F_{\nu\rho}g^{\lambda\rho} - \frac{1}{4}g_{\mu\nu}F_{\lambda\rho}F_{\sigma\tau}g^{\lambda\sigma}g^{\rho\tau},$$

the so-called *Reissner-Nordström* solution. Here the parameters are mass m and charge e. For $e = 0$ one retrieves the Schwarzschild family, while for $0 < e < m$ one obtains:
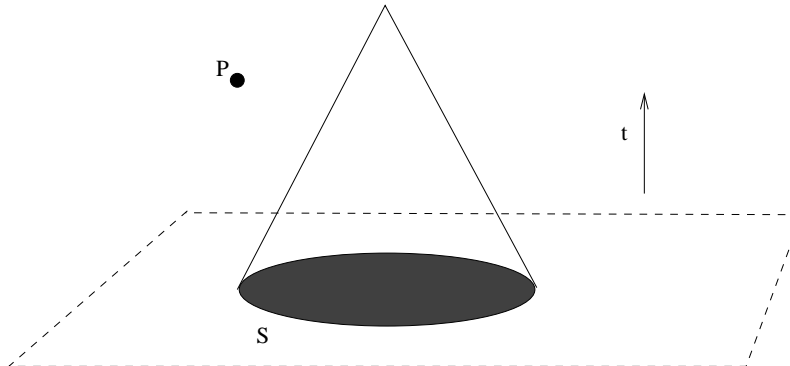

The $r = 0$ singularity of the Schwarzschild solution has disappeared! The above spacetime is completely regular up to the edges. These new edges that "complete" the triangle, however, are at a *finite* distance from the initial data, in the sense that all timelike geodesics joining those edges with the initial hypersurface have finite length. This solution thus has a regular future boundary and is extendible (in C^∞!) beyond it.

What fails at the boundary of this maximal domain of development of initial data is thus not the regularity of the solution, but rather, global hyperbolicity. Any extension of Q will contain past inextendible causal geodesics not intersecting the initial hypersurface:

Points in such an extension but not in Q itself cannot be determined by initial data, in the exact same way that the solution to a linear wave equation $\Box\Psi = 0$ at the point P depicted below, cannot be uniquely determined by its values in

the shaded set S:

What has happened in the Reissner-Nordström solution is that the situation depicted above developed, but from an S that was complete.

7   Strong Cosmic Censorship

The physical interpretation of the above situation is that the classical principle of determinism fails, but without any sort of loss in regularity that would indicate that the domain of the classical theory has been exited. It is in that sense that this kind of behavior is widely considered by physicists to be problematic[4]. On the other hand, numerical calculations (Penrose and Simpson [23]) on the behavior of linear equations on a Reissner-Nordström background indicate that a naturally defined derivative blows up at the Cauchy horizon. This was termed the blue-shift effect. Thus, Penrose argued, the pathological behavior of the Reissner-Nordström solution might be unstable to perturbation. This led him to conjecture, more generally,

Strong Cosmic Censorship For generic initial data, in an appropriate class, the maximal domain of development is inextendible.

In view of our discussion in the introduction, this can be thought of as the conjecture that, for generic initial data, the solution is unique wherever it can be defined. Of course, the context in which this should be applied to (i.e. what equations, what class of initial data should be considered, etc...) and the notion of extendibility are left open. We will comment more on that later.
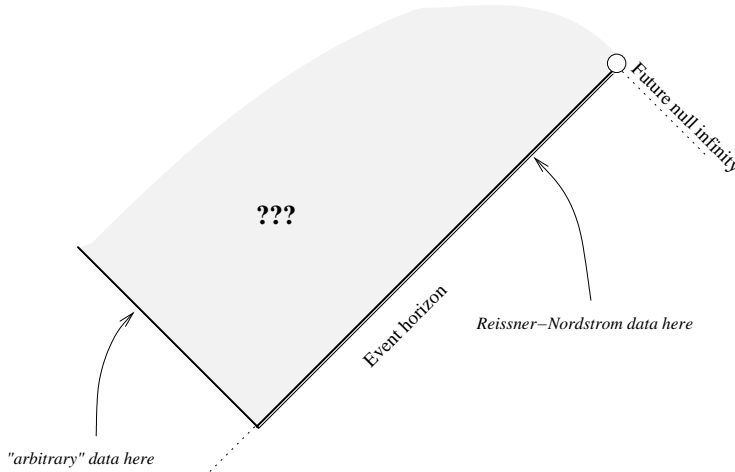
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[4]Of course, this failure of determinism only applies to observers who enter into black holes. In particular, Property 1 ensures that determinism holds in the domain of outer communications (hence the term "weak" cosmic censorship). On the other hand, it seems that fundamental principles of physics should be valid everywhere, including the interiors of black holes.

8 The Einstein-Maxwell-Scalar Field system

It might seem at first that the proper setting for discussing the problem of whether Cauchy horizons arise in evolution, for generic data in spherical symmetry, is the Einstein-Maxwell equations. Unfortunately, these suffer in fact from the same drawback as the Einstein vacuum equations, namely they do not possess the required dynamical degrees of freedom. One necessarily has to include more matter, and again the simplest choice, as in the work of Christodoulou, is a scalar field. Thus one is easily led to the coupled Einstein-Maxwell-Scalar Field system:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 2T_{\mu\nu} = 2(T^{em}_{\mu\nu} + T^{sf}_{\mu\nu})$$

$$F^{\mu\nu}_{;\nu} = 0,$$

$$F_{[\mu\nu,\rho]} = 0,$$

$$g^{\mu\nu}(\partial_\mu\phi)_{;\nu} = 0,$$

$$T^{em}_{\mu\nu} = F_{\mu\lambda}F_{\nu\rho}g^{\lambda\rho} - \frac{1}{4}g_{\mu\nu}F_{\lambda\rho}F_{\sigma\tau}g^{\lambda\sigma}g^{\rho\tau},$$

$$T^{sf}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi.$$

It turns out that in spherical symmetry the Maxwell part of the equation decouples. Since the scalar field ϕ carries no charge, a non-trivial Maxwell field can only be present if an initial complete spacelike hypersurface has non-trivial topology. In particular, this model is not suitable for considering the formation of black holes, as in the work of Christodoulou. Thus we will consider the problem where there is already a black hole present initially. To take the simplest possible formulation that captures the essense of the problem at hand, one can prescribe initial data for the system on two null rays, such that one corresponds to the event horizon of a Reissner-Nordström solution, and the other carries "arbitrary" matching data:

To make the most of the method of characteristics, we introduce null coordinates, i.e. coordinates (u, v) such that the metric on Q takes the form $-\Omega^2 du dv$. Here we select the v-axis to be the event horizon, and the u axis to be the conjugate ray on which we presribe our data. The unknowns are then just r, Ω and ϕ, and the electromagnetic part contributes a constant $e \neq 0$ which is computed from the initial data. To write the equations as a first order system, we define $\partial_u r = \nu$, $\partial_v r = \lambda$, $r \partial_u \phi = \zeta$, $r \partial_v r = \zeta$, and also ϖ, what one can call the "renormalized" Hawking mass[5], defined by

$$1 - \frac{2\varpi}{r} + \frac{e^2}{r^2} = |\nabla r|^2 = -4\Omega^{-2}\lambda\nu.$$

We then have:

$$\partial_u r = \nu, \tag{2}$$

$$\partial_v r = \lambda, \tag{3}$$

$$\partial_v \nu = \nu \left(-\frac{2\lambda}{1-\mu} \frac{1}{r^2} \left(\frac{e^2}{r} - \varpi \right) \right), \tag{4}$$

$$\partial_u \varpi = \frac{1}{2}(1-\mu) \left(\frac{\zeta}{\nu} \right)^2 \nu, \tag{5}$$

$$\partial_v \varpi = \frac{1}{2}(1-\mu) \left(\frac{\theta}{\lambda} \right)^2 \lambda, \tag{6}$$

$$\partial_u \theta = -\frac{\zeta\lambda}{r}, \tag{7}$$

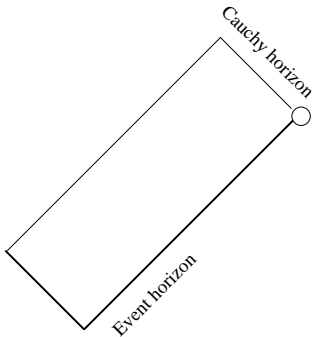$$\partial_v \zeta = -\frac{\theta\nu}{r}. \tag{8}$$

9   Statement of the theorems

We can now state the theorems of [16]. On the one hand we have:

Theorem 1 *After restricting the range of the u coordinate, the Penrose dia-*

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[5]The Hawking mass m is defined to be $m = \frac{r}{2}(1 - |\nabla r|^2)$. The renormalized version has the property that it is constant in the Reissner-Nordsröm solution and coincides with m at Future Null Infinity.

gram of the solution of the I.V.P. described above is as follows:

Moreover, r extends to a function on the Cauchy horizon with the property $r \to \varpi_{init} - \sqrt{\varpi_{init}^2 - e^2}$ *as* $u \to 0$ *(where* ϖ_{init} *is the constant value of* ϖ *on the initial* event *horizon), and the metric can be continuously extended globally across the Cauchy horizon.*

On the other hand, we have:

Theorem 2 *For "generic" initial data in the class of allowed initial data,* ϖ *blows up identically on the Cauchy horizon. In particular, the solution is inextendible across the Cauchy horizon as a* C^1 *metric.*

Thus, strong cosmic censorship is true, according to Theorem 2, if formulated with respect to extendibility in C^1 or higher, but false, according to Theorem 1, if formulated with respect to extendibility in C^0 (See [7] for reasons why one might want to require C^0.) In any case, the formation of a null "weak" singularity indicates a qualitatively different picture of the internal structure of the black hole from any of the previous models described above, and from the original expectations of Penrose.

The scenario of Theorems 1 and 2 was first suggested by Israel and Poisson [22] who put forth some heuristic arguments. Subsequently, a large class of numerics was done for precisely the equations considered here (see [3] for a survey). Because of the blow-up in the mass, this phenomenon was termed "mass inflation".

It should be noted that the imposition of Reissner-Nordström data on the event horizon is somewhat unnatural if the data is viewed as having arisen from generic data for a characteristic value problem where the v-axis is in the domain of outer communications[6]. Similar results to the above theorems, however, can in fact be proven for a wide class of data which includes the kind conjectured to arise from the aforementioned problem. These results will appear in [17]. The relevance of such an extension will only become clear, however, if the problem of

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[6]It should be emphasized that our rationale in choosing the I.V.P. here was to separate completely the issue of the dynamics of the interior of the black hole from the precise understanding of the set of data that arises in collapse, which is a problem of very different analytical flavor.

determining the correct "generic" decay on the event horizon is mathematically resolved.

10 Some ideas from the proofs

Our initial data are trapped, i.e. ν and λ are both nonpositive. These signs are then preserved in evolution. Note that from the equation

$$\partial_u \left(\frac{\lambda}{1-\mu} \right) = \left(\frac{\lambda}{1-\mu} \right) \frac{1}{r} \left(\frac{\zeta}{\nu} \right)^2 \nu \qquad (9)$$

it follows that $\frac{\lambda}{1-\mu}$ is non-increasing in u[7]. What gives the analysis of our equations *in the black hole region* its characteristic flavor is the fact that $\int \frac{\lambda}{1-\mu} dv$ is potentially infinite when integrated for constant u along the whole range of v (it is indeed infinite in initial data, i.e. $u = 0$), and that this infinity can appear in the equations (for instance in (4)) with either a positive or negative sign, depending on the sign of $\frac{e^2}{r} - \varpi$. Note that, by contrast, in the domain of outer communications, this infinity is killed by the $\frac{1}{r^2}$ term since $r \to \infty$ on outgoing rays. In the domain of development of our initial data, r is bounded above by its initial constant value on the event horizon $\varpi_{init} + \sqrt{\varpi_{init}^2 - e^2}$, in view of the signs of λ and ν.
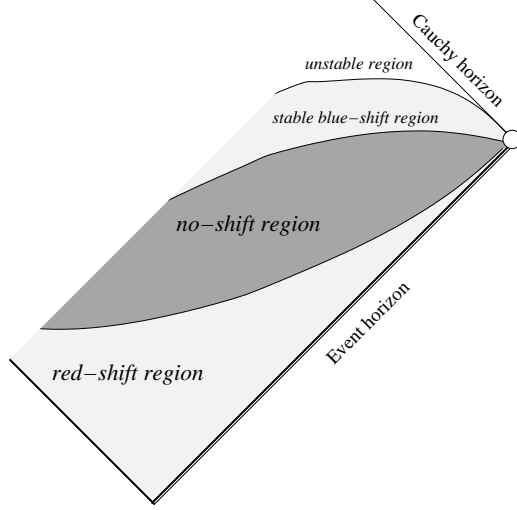
In the Reissner-Nordström solution, the sign of $\frac{e^2}{r} - \varpi$ goes from negative near the event horizon to positive near the Cauchy horizon, while $\int \frac{\lambda}{1-\mu} dv$ remains constant in u and thus infinite, accounting for both what is called the infinite red shift near the event horizon (this makes objects crossing the event horizon slowly disappear to outside observers as they are shifted to the red) and the infinite blue shift near the Cauchy horizon (this accounts for the instability of the Cauchy horizon to *linear* perturbations).

For general solutions of the initial value problem, some of these features of the sign of $\frac{e^2}{r} - \varpi$ turn out to be stable, while others do not. In particular, Theorems 1 and 2 together imply that the sign must become negative near the Cauchy horizon, and not positive! To attack this initial value problem, it is clear that the behavior of this sign is the first thing that must be understood. It turns out that before the effects of the linear instability start to play a role, three geometrically distinct regions develop in evolution, a red-shift, no-shift, and stable blue-shift region, characterized by

$$\frac{e^2}{r} - \varpi < -\epsilon \ , \ \frac{e^2}{r} - \varpi \sim 0 \ , \ \frac{e^2}{r} - \varpi > \epsilon,$$

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[7]That this quantity is non-increasing is in fact a general feature of spherical symmetry, i.e. it depends only on the dominant energy condition, not on the particular choice of matter.
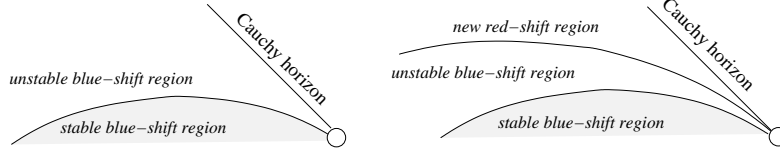
respectively:

In the red-shift region, $\int \frac{\lambda}{1-\mu} dv$ is unbounded as $u \to 0$, but it appears with a "favorable" sign (favorable as far as controlling ϖ is concerned[8]), in the no-shift region, $\int \frac{\lambda}{1-\mu}$ is uniformly bounded, and in the stable blue-shift region, $\int \frac{\lambda}{1-\mu} dv$ grows with u but at a rate "less" than the growth of certain natural derivative of ϕ. These facts allow us to control all quantities reasonably well up until the future boundary of the stable blue-shift region, though completely different arguments must be applied to each subsequent region as it develops in evolution from the previous one.

Of course, all this work seems only to have pushed forward the problem from the original initial segments to the future boundary of the "stable blue-shift region". But in fact our new "initial" conditions on the future boundary of the stable blue-shift region are much more favorable. The stable blue-shift region that has preceded it ensures that ν has a sufficiently fast decay rate in u. (Remember that blue-shift regions tend to make $|\nu|$ smaller, so they are favorable for controlling r, but unfavorable for controlling ϖ.) Once this rate can be shown to be preserved, it follows by integrating u that one can bound r *a priori* away from 0 in its future, and thus prove the existence of the solution up to the Cauchy horizon. It is clear from what we have said above that if the unstable region remains a blue-shift region (see left diagram below), there is no problem. (This is of course what happens for the Reissner-Nordsröm solution

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[8]This sign tends to make ν bigger, and there is an extra ν on the denominator of (5).

itself.)

The danger is if a new red-shift region develops (see right diagram above). It turns out, however, that a new *a priori* estimate

$$\int \partial_v \log(\nu) dv = -\int \frac{2\lambda}{1-\mu}\frac{1}{r^2}\left(\frac{e^2}{r} - \varpi\right) dv < C$$

is available in this case, which is independent of the size of ϖ. This makes use of the fact that there is a ϖ hidden also in the denominator in $1 - \mu$. (The estimate depends, however, on our knowledge of the new initial condition of r and a particular bootstrap assumption on its future behavior; in particular this estimate does not hold in the original "red-shift" region.) This is the last element in the proof of Theorem 1.

We now discuss the proof of Theorem 2. As has been noted, at the level of perturbation theory [4], one does see an instability, caused by the blue-shift region. (Requiring some derivative of the scalar field to be positive initially, ζ will decay in u, and thus θ will decay in v, at a slower rate than the decay of λ in v, so that the natural derivative $\frac{\theta}{\lambda} \to \infty$.) On the other hand, the non-linearity tends to diminish this effect, since if the mass indeed increases, in view of Theorem 1, we have a reappearance of a red-shift region (see diagram on the right above). As remarked earlier, this tends to make $|\nu|$ (and also $|\lambda|$) bigger, and thus $\left|\frac{\theta}{\lambda}\right|$ smaller. The proof must thus encorporate something beyond "linear theory arguments".

This extra ingredient is supplied by a very powerful monotonicity peculiar to black hole interiors, or more specifically, to "trapped regions". Integration of (7) and (8) then implies that if θ and ζ are initially of the same signs[9], let's say non-negative, then they remain non-negative, and in fact $\partial_v \zeta \geq 0$ and $\partial_u \theta \geq 0$. Now integration of (6) and using that $\frac{1-\mu}{\lambda}$ is also non-increasing in u, yields that for $v_2 > v_1$, $u_2 > u_1$,

$$\varpi(u_2, v_2) - \varpi(u_2, v_1) \geq \varpi(u_1, v_2) - \varpi(u_1, v_1). \tag{10}$$

It should be mentioned that in view of the sign of λ, both sides of the above inequality are positive.

The broad outline of the proof of Theorem 2 is as follows: First assume that the spacetime looks very much like Reissner-Nordström. Then the linear theory
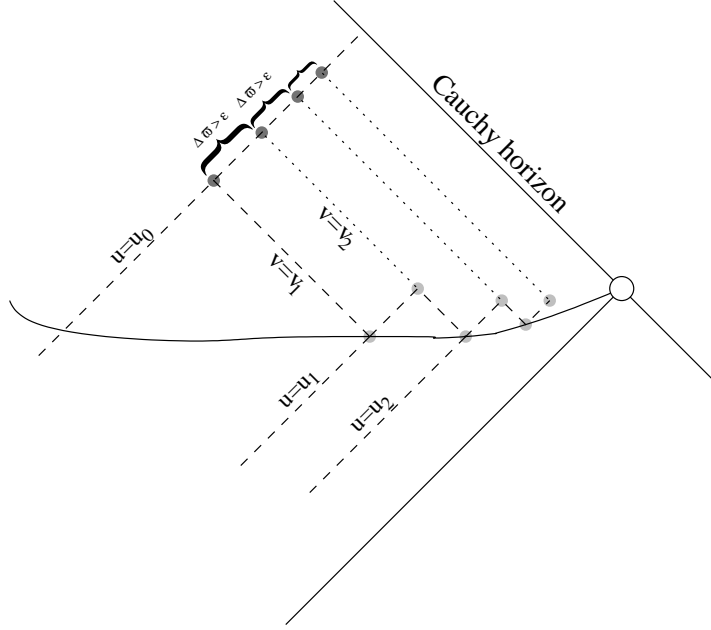
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[9]In view of the fact that the data vanish on the event horizon, this can be considered a generic condition after restricting the domain of the u coordinate. This condition and the non-vanishing of a particular derivative of ϕ at the origin together define the "generic" class of initial data to which Theorem 2 applies.

more or less applies, and applying the bounds for θ in the equation (6), one obtains $\varpi \to \infty$, which is a contradiction. Thus one is reduced to proving that any spacetime "quantitatively different" from Reissner-Nordström must have $\varpi \to \infty$.

It is not possible here to explain precisely what "quantitatively different" has to mean. To give a taste of the kind of arguments involved in this "non-linear part" of the proof, we will be content to show that assuming only that ϖ is bounded below by a positive number plus its Reissner-Nordström value (this is indeed a quantitative difference)

$$\varpi_{Cauchy\ horizon} > \varpi_{init} + 2\epsilon, \tag{11}$$

it follows from (10) that ϖ must in fact blow up identically on the Cauchy horizon. If γ is the future boundary of the stable blue-shift region, the fact that $\varpi_\gamma(u) \to \varpi_{init}$ implies that given any u_0, a sequence of points (u_i, v_i) can be constructed so that the mass differences $\varpi(v_{i+1}, u_i) - \varpi(v_i, u_i)$ are all greater than ϵ:

But in view of (10), this mass difference can be added on $u = u_0$ to yield infinite mass at the point where $u = u_0$ meets the Cauchy horizon.

Suffice it to say that the non-linear analysis of black hole regions is quite different than the analysis we are used to. Which parts of this spherically-symmetric picture generalize and which do not remains to be seen.

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